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Definitions

1. State the set-builder definition of $\text{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

2. Define linear **Independence** of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

3. Define "$T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation"

4. Define "$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one"

5. Define "$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto"

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General Linear Transformations

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 3 \\ 1 & -4 & -1 \end{bmatrix}, \qquad \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \vec{\mathbf{b}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \vec{\mathbf{c}} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

   (a) Is $\vec{\mathbf{u}}$ a solution to $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$?

   (b) Find all solutions to the equation $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$

   (c) Is $\vec{\mathbf{c}}$ in the range of T? Justify your answer.

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2. Let Tbe a transformation defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ -3 & 2 \end{bmatrix}$

   (a) What is the domain of T?

   (b) What is the co-domain of T?

   (c) Prove that the Range of T as the span of a set of vectors by rewriting the set builder notation.

   (d) Sketch the general form of T as a function from the domain to the co-domain. Use a square to represent $\mathbb{R}^2$ and a cube to represent $\mathbb{R}^3$.

3. (No Computation) How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^5$ into $\mathbb{R}^7$ by the rule $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$?

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4. Let $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{\mathbf{v}}_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{\mathbf{x}}$ to $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2$. Find a matrix A so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ for every $\vec{\mathbf{x}}$.

5. Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that sends $\vec{\mathbf{e}}_1$ to $\vec{\mathbf{e}}_1 - 3\vec{\mathbf{e}}_2$ and leaves $\vec{\mathbf{e}}_2$ unchanged.

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6. Suppose that $A = \left[\vec{\mathbf{a}}_1 \dots \vec{\mathbf{a}}_n\right]$ and $B = \left[\vec{\mathbf{b}}_1 \dots \vec{\mathbf{b}}_n\right]$ are $m \times n$ matrices, and that $\vec{\mathbf{x}} \in \mathbb{R}^n$.

   Prove that $(A + B)\vec{\mathbf{x}} = A\vec{\mathbf{x}} + B\vec{\mathbf{x}}$ from first principles.

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7. Define $T : \mathbb{R}^1 \to \mathbb{R}^1$ by $T(\vec{\mathbf{x}}) = 3\vec{\mathbf{x}}$.

   Prove that T is a linear transformation.

8. Define $T : \mathbb{R}^1 \to \mathbb{R}^1$ by $T(\vec{\mathbf{x}}) = 3\vec{\mathbf{x}} + \vec{\mathbf{1}}$, where $\vec{\mathbf{1}} = [1]$.

   Prove that T is not a linear transformation.

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9. Let transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with

$$T(\vec{\mathbf{e}}_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } T(\vec{\mathbf{e}}_2) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}.$$

(a) Find the standard matrix A of T. Then sketch the general form of T.

(b) Determine if the transformation T is one-to-one.

(c) Determine if the transformation T is onto.

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10. Let transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with

$$T(\vec{\mathbf{e}}_1) = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, T(\vec{\mathbf{e}}_2) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \text{ and } T(\vec{\mathbf{e}}_3) = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

(a) Find the standard matrix A of T. Then sketch the general form of T.

(b) Determine if the transformation T is one-to-one.

(c) Determine if the transformation T is onto.

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11. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with standard form matrix A.

    Prove that T *cannot* be onto. (You must give a *complete* argument).

12. Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that *is* one-to-one (Hint: define T by choosing its standard matrix).

13. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is linear. Is it true that T is one-to-one if and only if T is onto? Why doesn't this violate the invertible matrix theorem?

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14. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with standard form matrix A.

    Prove that T *cannot* be one-to-one. (You must give a *complete* argument).

15. Give an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ that *is* onto (Hint: define T by choosing its standard matrix).

16. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear. Is it true that T is one-to-one if and only if T is onto? Why doesn't this violate the invertible matrix theorem?

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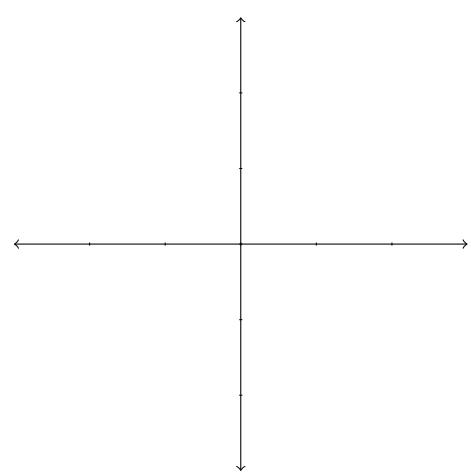
17. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where

$$A = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

(a) Compute where T sends the following vectors.

- $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) =$
- $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) =$
- $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) =$
- $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) =$

(b) Sketch where T sends the box with corners $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



(c) Describe geometrically how the T transforms $\mathbb{R}^2$

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Matrix Operations

1. Compute the following matrix operations, or explain why they are undefined.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix}$$

(a) AB

(b) BA

(c) A^T

(d) $2A - 3B$

(e) $2A^T - 3B$

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2. Let

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

(a) What is the domain and co-domain of the transformation T defined by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$? Sketch the general form of T.

(b) Compute A^{-1}.

(c) Let $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$. Use the inverse of A to solve the matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

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3. Rewrite $(AB)^{-1}(B+A)$ using the properties of matrix operations.

   (Careful: Multiplication is *not* commutative).

4. Rewrite $(B+A)(AB)^{-1}$ using the properties of matrix operations.

   (Careful: Multiplication is *not* commutative).

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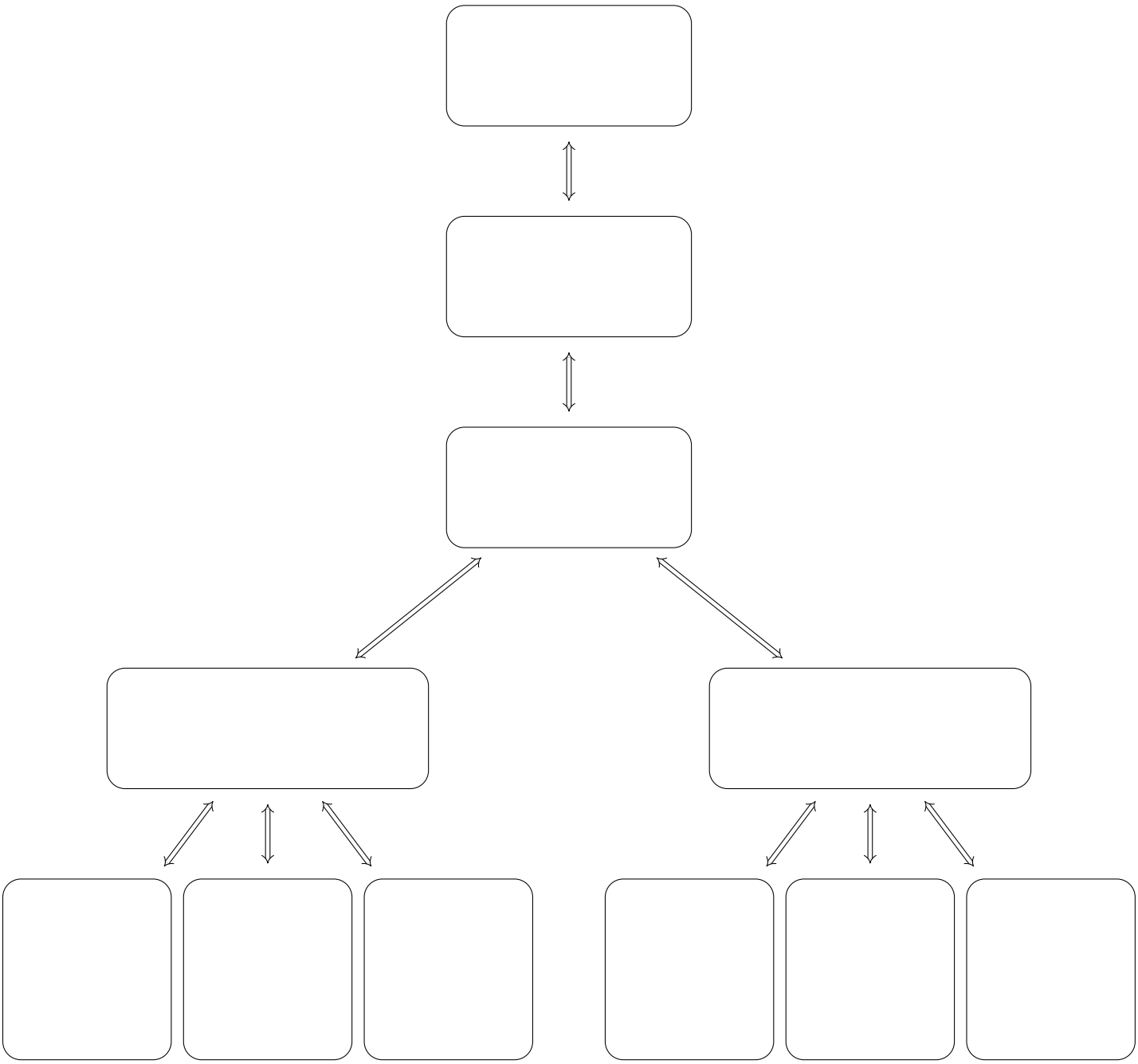
Square Matrices

The Invertible Matrix Theorem (IMT)

Suppose that A is an $n \times n$ matrix, and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

Then, the following are equivalent.



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1. Suppose that A is an $n \times n$ matrix. Prove the following statements by "walking through the tree" of the Invertible Matrix Theorem. **You must show every step**.

   (a) Suppose that an $n \times n$ matrix A is invertible.
   Prove that the columns of A span $\mathbb{R}^n$.

   (b) Suppose that an $n \times n$ matrix A is not invertible. Prove that the columns of A are linearly dependent.

   (c) Suppose that A is an $n \times n$ matrix, and that $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a unique solution. Prove that A is invertible.

   (d) Suppose that A is an $n \times n$ matrix, and that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ does not have a solution for some $\vec{\mathbf{b}}$ in $\mathbb{R}^n$. Prove that A is not invertible.

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2. Determine if the following matrices are invertible using as few computations as possible.

   (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

   (b) $A = \begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

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Basic Theorems

Theorem 2 There are 3 cases for the reduced echelon form of a linear system's augmented matrix

1. The system has _____ solutions if _____
2. The system has _____ solutions if _____
3. The system has _____ solutions if _____

Theorem 4: The columns of an $m \times n$ matrix A span $\mathbb{R}^m$

if and only if there is a pivot _____

Shortcuts to Recognize Dependence

- *If* one column of A is a multiple of another, *then* the columns of A are linearly dependent.
- *If* $\{\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n\}$ contains $\vec{\mathbf{0}}$, *then* $\{\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n\}$ is linearly dependent.
- *If* an $m \times n$ matrix A has more columns than rows (if $n > m$), *then* the columns of A are linearly dependent.

Theorem 5 If A is an $m \times n$ matrix, $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, *Then*

- $A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) =$ _____
- $A(c \cdot \vec{\mathbf{u}}) =$ _____

Properties of Linear Transformations

- If T is linear, then $T(\vec{\mathbf{0}}) =$ _____
- T is linear $\iff$ $T(c \cdot \vec{\mathbf{u}} + d \cdot \vec{\mathbf{v}}) =$ _____

Theorem 10 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear.

Then there is a unique $m \times n$ matrix A s.t. $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

In Fact, $A =$ _____

Theorem 12 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear with standard matrix A. T

(a) T is onto $\iff$ _____ $\iff$ _____

(b) T is one-to-one $\iff$ _____ $\iff$ _____

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Determinants

1. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Compute $det(A)$.

2. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$. Compute $det(A)$.

3. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 5 \\ 2 & 6 & 0 \end{bmatrix}$. Compute $det(A)$.

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4. Find k so that $A = \begin{bmatrix} 1 & k & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix}$ is invertible.

5. Find k so that $A = \begin{bmatrix} 0 & k & 2 \\ 3 & 0 & k \\ 1 & k & 0 \end{bmatrix}$ is invertible.

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6. Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$. Compute $det(A)$.

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7. You can compute the determinant of a large matrix by *reducing to echelon form* using only interchange and replacement, tracking how the determinant changes at each step.

   (a) Interchange _____

$$det \begin{bmatrix} — \mathbf{r}_1 — \\ — \mathbf{r}_2 — \\ — \mathbf{r}_3 — \end{bmatrix} = (-1) \cdot det \begin{bmatrix} — \mathbf{r}_2 — \\ — \mathbf{r}_1 — \\ — \mathbf{r}_3 — \end{bmatrix}$$

   (b) Replacement _____

$$det \begin{bmatrix} — \mathbf{r}_1 — \\ — \mathbf{r}_2 — \\ — \mathbf{r}_3 — \end{bmatrix} = det \begin{bmatrix} — \mathbf{r}_1 — \\ — \mathbf{r}_2 + 3\mathbf{r}_1 — \\ — \mathbf{r}_3 — \end{bmatrix}$$

   (c) Rescaling: _____

$$det \begin{bmatrix} — \mathbf{r}_1 — \\ — 2 \cdot \mathbf{r}_2 — \\ — \mathbf{r}_3 — \end{bmatrix} = (2) \cdot det \begin{bmatrix} — \mathbf{r}_1 — \\ — \mathbf{r}_2 — \\ — \mathbf{r}_3 — \end{bmatrix}$$

8. Let $A = \begin{bmatrix} 1 & 4 & 2 & 4 \\ 2 & 8 & 4 & 10 \\ 0 & 4 & 6 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$. Compute $det(A)$.